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Approximating pseudopotentials, scattering problems, and Bäcklund transformations for the Korteweg–de Vries and modified Korteweg–de Vries equations with perturbations

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Abstract. Approximating pseudopotentials associated with scattering problems and with Bäcklund transformations are derived for the perturbed Korteweg–de Vries and modified Korteweg–de Vries equations. Recursion formulas for approximate conservation laws are obtained for the perturbed Korteweg–de Vries equation as well. As an example, the auto-Bäcklund transformations are used to construct approximate one and two-soliton solutions.

1. Introduction

It is known that integrable models describing physical processes, such as the Korteweg–de Vries (KdV) and modified Korteweg–de Vries (MKdV) equations, are derived as the leading-order approximations of the original 'non-integrable' models. In doing so terms with a small parameter are not taken into account. However, as a rule, these terms are necessary for the comprehensive description of the phenomena under consideration. The existence of such terms in partial differential equations is caused by weak nonlinearities, dissipation, etc. In order to find solutions for such equations that correctly describe physical systems of interest, there is no need to employ exact methods. In fact, any exact solution would be only an approximation to a real process and therefore the use of an approximation is more than sufficient.

The pseudopotential technique, which allows one to construct exact Bäcklund transformations, scattering problems, etc, is well known for integrable nonlinear equations [1]. In the present work its generalization for non-integrable ones with a small parameter [2] is applied to find the approximate recursion formulas for conservation laws, Bäcklund transformations, and scattering technique for the perturbed KdV and MKdV equations.

The following definition of approximating pseudopotentials was proposed [2]. Let us take an evolution differential equation with a small parameter

$$u_t = K(x, t, u, u_x, \ldots, u_{nx}, \varepsilon) \qquad n \in N \qquad |\varepsilon| \ll 1. \tag{1.1}$$

A function $q(x, t)$ (vectorial in general) is called an approximating pseudopotential (AP) of the mth order for (1.1) if it is determined by a pair of equations

$$\begin{aligned} q_x &= P(x, t, v, \ldots, v_{1x}; q; \varepsilon) \\ q_t &= Q(x, t, v, \ldots, v_{1x}; q; \varepsilon) \qquad 1 \in N \qquad v = v(x, t) \end{aligned} \tag{1.2}$$

whose compatibility condition

$$q_{xt}-q_{tx}=0 \tag{1.3}$$

can be presented as

$$L(v_t-K(x, t, v, \ldots, v_{nx}, \varepsilon))+O(\varepsilon^{m+1})=0 \qquad m\in N \tag{1.4}$$

where L is a linear differential operator with vectorial coefficients.

The function v is an approximation for u in the sense that v_i $(i=\overline{0, m})$ in an asymptotic series

$$v=v_0+\varepsilon v_1+\varepsilon^2 v_2+\ldots$$

satisfy the same equations

$$K_j=0 \qquad j=\overline{0, m} \tag{1.5}$$

as in a direct perturbations approach. Employing the method of multiple scales, the operator L is replaced by the expansion

$$L=L_0+\varepsilon L_1+\ldots$$

and v_t-K is expended as

$$v_t-K=K_0+\varepsilon K_1+\varepsilon^2 K_2+\ldots.$$

Then equating the coefficients of each order in ε up to ε^m to zero in (1.4), one obtains

$$L_0K_0=0$$

$$L_0K_i+\sum_{j=0}^{i-1} L_{i-j}K_j=0 \qquad i=\overline{1, m}.$$

For these relations to be fulfilled, it is sufficient that equations (1.5) are satisfied. The use of the direct method for (1.2) leads to a set of the exact linear pseudopotentials of (1.5), and this allows one to apply such exact pseudopotentials for obtaining approximate solutions in case q is an unbounded function.

In the following we assume that P and Q do not depend explicitly on x and t and that P depends only on v, q and ε.

The general procedure for finding P and Q with

$$P(v, q; \varepsilon)=P_0+\varepsilon P_1+\ldots+\varepsilon^m P_m+\varepsilon^{m+1}R_1$$

$$Q(v, \ldots, v_{(n-1)x}, q; \varepsilon)=Q_0+\varepsilon Q_1+\ldots+\varepsilon^m Q_m+\varepsilon^{m+1}R_2$$

$$R_1, R_2\simeq O(1)$$

now consists in solving consecutively the recursion relations [2]

$$\begin{gathered}\sum_{j=0}^{i}\left(\frac{\partial P_j}{\partial v}F_{i-j}+[P_j, Q_{i-j}]\right)-\sum_{j=0}^{n-1} v_{(j+1)x}\left(\frac{\partial}{\partial v_{jx}}Q_i\right)=0 \qquad i=0, 1, 2\ldots \\ F_j=\frac{1}{j!}\left.\frac{\partial^j K}{\partial\varepsilon^j}\right|_{\varepsilon=0} \qquad \forall j \qquad [P, Q]=Q\frac{\partial}{\partial q}P-P\frac{\partial}{\partial q}Q\end{gathered} \tag{1.6}$$

in the same manner as proposed in [3, 4] and in determining R_1, R_2. Since the relations to be solved for P_i, Q_i $(i>0)$ are linear inhomogeneous equations with right-hand parts depending on the perturbation, arbitrary constants appear in their solutions.

Assume that in (1.2) $\boldsymbol{P}$ and $\boldsymbol{Q}$ are of the form

$$\boldsymbol{P}=P(v;\lambda;\varepsilon)\boldsymbol{q} \tag{1.7}$$

$$\begin{aligned}\boldsymbol{Q}&=Q(v,\ldots,v_{lx};\lambda;\varepsilon)\boldsymbol{q}+\boldsymbol{T}(x,t;\boldsymbol{q};\varepsilon)\\ \boldsymbol{T}&\simeq O(\varepsilon^{m+1})\end{aligned} \tag{1.8}$$

where P and Q are some matrices depending rationally on the spectral parameter λ [5]. As equations (1.7) and (1.8) are linear up to ε^m order, the inverse scattering transform can immediately be used to solve (1.1) approximately. However, applying a technique analogous to that proposed and developed in [6, 7], these APs may also be used to find higher-order corrections to an approximate solution.

Taking into account (1.4), (1.7) and (1.8), let us write (1.1) as

$$\begin{aligned}&\frac{\partial}{\partial t}P-\frac{\partial}{\partial x}Q+[P,Q]+\varepsilon^{m+1}R=0\\ &R=-\varepsilon^{-m-1}\left(\sum_{i+j>m}\varepsilon^{i+j}\left(\frac{\partial P_j}{\partial v}F_i+[P_j,Q_i]\right)-\sum_{j>m}\varepsilon^{j}\sum_{k=0}^{l}v_{(k+1)x}\frac{\partial}{\partial v_{kx}}Q_j\right)\end{aligned} \tag{1.9}$$

([,] denotes the matrix commutator.) Differentiating (1.7) with respect to t and taking into consideration (1.9), one has

$$\boldsymbol{q}_{tx}=P\boldsymbol{q}_t+\left(\frac{\partial}{\partial x}Q-[P,Q]\right)\boldsymbol{q}-\varepsilon^{m+1}R\boldsymbol{q}.$$

Then inserting (1.8) into this equation, one obtains

$$\boldsymbol{T}_x-P\boldsymbol{T}=-\varepsilon^{m+1}R\boldsymbol{q}.$$

This inhomogeneous form of (1.7) can be solved by the method of variation of parameters

$$\boldsymbol{T}=-\varepsilon^{m+1}\Phi(x)\int_{-\infty}^{x}\Phi^{-1}(\tau)R\boldsymbol{q}\,\mathrm{d}\tau$$

where $\Phi(x)$ is the fundamental system of solutions for (1.7). To obtain higher-order corrections, one needs to take into account the expression for $\boldsymbol{T}$ in (1.7) and (1.8) and employ iterations [6, 7].

2. The approximating pseudopotentials and approximate solutions of the perturbed KdV equation

In the description of weakly nonlinear, long wavelength waves propagating on the surface of an incompressible, inviscid, irrotational fluid the following perturbed KdV equation (PKdV) arises [8]

$$u_t+6uu_x+u_{xxx}+\varepsilon(-\alpha u^2u_x+\beta uu_{xxx}+\gamma u_xu_{xx}+\delta u_{xxxxx})=0 \qquad |\varepsilon|\ll 1. \tag{2.1}$$

Here α, β, γ, δ are constant parameters; ε is the amplitude-to-depth ratio. The derivation of this equation, computer simulation, and experimental results for (2.1) may be found in [8]. Approximate soliton and cnoidal waves are also presented there.

In principle, equation (2.1) can be transformed into a higher-order KdV equation [9] by means of a so-called canonical map up to order ε [10, 11]. The corresponding nonlinear integro-differential operator of the second order can be applied to find some special approximate solutions of (2.1), but its use together with a Lax pair of the higher-order KdV for solving an initial problem is restricted because of integration and the necessity to restore a function $u(x, t)$. In addition, that transformation maps the hierarchy of local conservation laws for the higher-order KdV into approximate non-local conservation laws, while a hierarchy of local ones, which is important from the viewpoint of physical applications, remains unknown.

In the present work for the study of equation (2.1), we use APs that lead to a scattering problem and Bäcklund transformations. As an example, the one and two-soliton solutions of (2.1) are constructed up to order ε. The recursion formulas for approximate local conservation laws are also derived. We shall consider the simplest AP of the first order

$$q_x = P_0(v, q) + \varepsilon P_1(v, q)$$

$$q_t = Q_0(v, v_x, v_{xx}, q) + \varepsilon Q_1(v, \ldots, v_{xxxx}, q)$$

choosing as P_0, Q_0 the following expressions, which correspond to a pseudopotential of the KdV equation and give rise both to the Lax pair [12] and to auto-Bäcklund transformation [13]

$$P_0 = (\lambda + v)E - F$$

$$Q_0 = (4\lambda^2 + 2\lambda v - 2v^2 - v_{xx})E + 2(v - 2\lambda)F - v_x H.$$

Here E, F, H are as follows

$$H = 2q \qquad E = 1 \qquad F = -q^2 \tag{2.2}$$

and form a representation of the Lie-algebra $sl(2)$ under the bracket of formula (1.6).

In appendix 1 a general form and particular solutions for P_1 and Q_1 are presented. In particular, taking into account (A1.1) and (A1.2), the above AP can be of the type

$$\begin{aligned} q_x = q^2 + \lambda + v + \varepsilon(&\lambda q^2(\alpha - 2\beta + \gamma + 30\delta) + q^2 v(-\alpha - 2\beta - \gamma + 10\delta) + 6cq^2 + 6\lambda hq \\ &+ 6hvq + 6qb - 6\lambda c + v^2(-3\beta + \gamma + 10\delta) - 6cv + 6a)/6 \end{aligned} \tag{2.3}$$

$$\begin{aligned} q_t = 4\lambda q^2 - 2q^2 v - 2qv_x + 4\lambda^2 + 2\lambda v - 2v^2 - v_{xx} + \varepsilon(&8q^2\lambda^2(\alpha - 2\beta + \gamma + 18\delta) \\ &+ 2\lambda q^2 v(-3\alpha + 2\beta - 3\gamma - 26\delta) + 24\lambda cq^2 + 2q^2v^2(2\alpha + 4\beta + \gamma - 18\delta) \\ &- 12cq^2 v - 6hq^2 v_x + q^2 v_{xx}(\alpha + 2\beta + \gamma - 22\delta) + 24q^2 a + 24\lambda^2 hq \\ &+ 12\lambda hqv + 4\lambda qv_x(-\alpha - \gamma + 2\delta) + 24\lambda qb - 12hqv^2 \\ &+ 4qvv_x(-\gamma + 2\delta) - 12qvb - 6hqv_{xx} - 12\delta qv_{xxx} \\ &+ 4\lambda^3(\alpha - 2\beta + \gamma + 6\delta) + 4\lambda^2 v(\alpha + \gamma - 2\delta) - 24\lambda^2 c \\ &+ 2\lambda v^2(\alpha + 2\gamma - 4\delta) - 12\lambda cv + 12\lambda\delta v_{xx} + 48\lambda a + 2v^3(\alpha + 5\beta - \gamma - 18\delta) \\ &+ 12cv^2 + 2vv_{xx}(-\gamma - 4\delta) + 12av + 2v_x^2(-\gamma + 2\delta) \\ &- 6v_x b + 6cv_{xx} - 6\delta v_{xxxx})/6 \qquad a, b, c, h = \text{constant}. \end{aligned} \tag{2.4}$$

Since the Riccati equations can be linearized, the form of these equations permits us to use them for deriving a Lax pair and auto-Bäcklund transformation. Following [12] and inserting $q=(\ln\psi)_x$ into (2.3) and (2.4), one has the approximate Lax pair for equation (2.1):

$$-\psi_{xx}+\varepsilon((-\alpha-2\beta-\gamma+10\delta)v_x/6+hv+b+h\lambda)\psi_x$$
$$+(\varepsilon(-a+(\alpha+5\beta-20\delta)v^2/6+2\lambda(\beta-10\delta)v/3$$
$$+\lambda^2(-\alpha+2\beta-\gamma-30\delta)/6)-v-\lambda)\psi=0 \tag{2.5}$$

$$-\psi_t+(\varepsilon(-6hv_x+2(\alpha+2\beta-8\delta)v^2+8\lambda(\beta-4\delta)v$$
$$+v_{xx}(\alpha+2\beta+\gamma-22\delta)+24a+4\lambda^2(\alpha-2\beta+\gamma+6\delta))/6$$
$$-2(v-2\lambda))\psi_x+((\varepsilon(2(\gamma-2\delta)v_xv+2\lambda(\alpha+\gamma-2\delta)v_x-3hv^2$$
$$+6(b-h\lambda)v+6\delta v_{xxx})/6+v_x)\psi=0 \tag{2.6}$$

(λ is the spectral parameter, and the terms of the order $O(\varepsilon^2)$ in the coefficients of ψ_x, ψ are missing). Clearly, at $\varepsilon=0$, (2.5) and (2.6) are transformed into the well-known Lax pair of the KdV equation [9, 13].

Equations (2.5) and (2.6) can be used to solve approximately the initial-value problems of equation (2.1). In principle, the constants in (2.5) may be chosen in such a way as to bring the spectral problem to the well-known type. For instance, letting $b, h=0$ for the case $\beta=10\delta$ and $\alpha+\gamma=-10\delta$, one arrives at the following problem, which differs from the KdV spectral problem [9, 13] by the type of potential

$$\psi_{xx}+(\varepsilon a+(\varepsilon/6)(\gamma-20\delta)v^2+v+\lambda)\psi=0.$$

The Lax pair (2.5) and (2.6) permits one to construct the N-soliton solutions of equation (2.1). However, as an example for this we shall use an auto-Bäcklund transformation that will be obtained below.

First of all, let us consider an evolution equation for q. For this purpose it is necessary to solve equation (2.3) with respect to v and eliminate it from (2.4).

Representing v by the expansion

$$v(x,t)=v_0(x,t)+\varepsilon v_1(x,t)+\ldots$$

and substituting it into (2.3), one arrives at the recursion relations for v_i ($i=0,1,\ldots$). (It should be kept in mind that these functions are not identical with the KdV solution and first-order correction, because q must also be expanded into an asymptotic series for this). In order to obtain the equation for q up to order ε

$$q_t=E_0(q,\ldots,q_{xxx})+\varepsilon E_1(q,\ldots,q_{xxxxx})+O(\varepsilon^2) \tag{2.7}$$

it is sufficient to find v_0 and v_1. One has

$$v_0=q_x-q^2-\lambda$$

$$v_1=((3\beta-\gamma-10\delta)q_x^2+(\alpha-4\beta+3\gamma+10\delta)q_xq^2-6hq_xq+2(3c-3\lambda\beta+\lambda\gamma+10\lambda\delta)q_x$$
$$+(-\alpha+\beta-2\gamma)q^4+6hq^3+2(-6c-\lambda\alpha+3\lambda\beta-2\lambda\gamma-20\lambda\delta)q^2-6bq$$
$$-6a+\lambda^2(3\beta-\gamma-10\delta))/6$$

and accordingly

$$E_0 = q_{xxx} - 6q_x q^2 - 6\lambda q_x$$

$$\begin{aligned} E_1 = {} & \delta q_{xxxxx} + (\beta - 10\delta) q_{xxx} q_x - \beta q_{xxx} q^2 - \lambda\beta q_{xxx} + (\beta - 10\delta) q_{xx}^2 \\ & + (\alpha - 2\beta + \gamma - 10\delta) q_{xx} q_x q \\ & - 3h q_{xx} q_x - \beta q_x^3 + (-2\alpha + \beta - 2\gamma) q_x q^4 + 6h q_x q^3 \\ & + 2(-6c - 2\lambda\alpha + 3\lambda\beta - 2\lambda\gamma - 20\lambda\delta) q_x q^2 - 6b q_x q \\ & + (-6a - \lambda^2\alpha + 3\lambda^2\beta - \lambda^2\gamma - 10\lambda^2\delta) q_x. \end{aligned}$$

At $\varepsilon = 0$ equation (2.7) is identical to the MKdV equation, which is invariant with respect to $q \to -q$. This fact allows one to use the corresponding exact pseudopotential as an auto-Bäcklund transformation for the unperturbed KdV equation [13]. At $\varepsilon \neq 0$ letting $b, h = 0$ without loss of generality, (2.7) is approximately invariant with respect to the more general substitution (see, e.g. [14] or a technique established in [15])

$$q \to -q - (2/3)\varepsilon(\beta - 10\delta)(q^2)_x \tag{2.8}$$

and equations (2.3) and (2.4) approximately map (2.1) into itself in the following way.

Assume that for the PKdV equation

$$u_t = F_0(u, \ldots, u_{xxx}) + \varepsilon F_1(u, \ldots, u_{xxxxx})$$

one has the AP

$$q_x = P_0(v, q) + \varepsilon P_1(v, q)$$

$$q_t = Q_0(v, \ldots, v_{xx}; q) + \varepsilon Q_1(v, \ldots, v_{xxxx}; q)$$

that determines the mapping $v \to q$ of a function $v(x, t)$ satisfying the equation

$$v_t = F_0(v, \ldots) + \varepsilon F_1(v, \ldots) + \varepsilon^2 S(v, \ldots; q; \varepsilon) \qquad S \simeq O(1) \tag{2.9}$$

into a function q satisfying the equation

$$q_t = E_0(q, \ldots) + \varepsilon E_1(q, \ldots) + \varepsilon^2 U(q, \ldots; \varepsilon) \qquad U \simeq O(1). \tag{2.10}$$

Further we have for the PKdV equation above also the AP

$$\tilde{q}_x = P_0(\tilde{v}, \tilde{q}) + \varepsilon P_1(\tilde{v}, \tilde{q})$$

$$\tilde{q}_t = Q_0(\tilde{v}, \ldots, \tilde{v}_{xx}; \tilde{q}) + \varepsilon Q_1(\tilde{v}, \ldots, \tilde{v}_{xxxx}; \tilde{q})$$

$$\tilde{q} = -q - (2/3)\varepsilon(\beta - 10\delta)(q^2)_x + O(\varepsilon^2)$$

determining the mapping $\tilde{v} \to \tilde{q}$ such that

$$\begin{aligned} \tilde{v}_t &= F_0(\tilde{v}, \ldots) + \varepsilon F_1(\tilde{v}, \ldots) + \varepsilon^2 S(\tilde{v}, \ldots; \tilde{q}; \varepsilon) \\ \tilde{q}_t &= E_0(\tilde{q}, \ldots) + \varepsilon E_1(\tilde{q}, \ldots) + \varepsilon^2 U(\tilde{q}, \ldots; \varepsilon) + O(\varepsilon^3). \end{aligned} \tag{2.11}$$

Equations (2.9)–(2.11) are identical to the PKdV or (2.7) in the first-order approximation. Therefore the functions v, $\tilde{v}$ and q, $\tilde{q}$ are approximate solutions for these equations in the above-defined sense. In consequence of this equations (2.3) and (2.4) and transformation (2.8) may be interpreted as the mapping

$$u \simeq v \to q \to \tilde{q} \to \tilde{v} \simeq \tilde{u}$$

and be used as the approximate auto-Bäcklund transformation. (For the KdV equation at $\varepsilon = 0$ we have the mapping $u = v \to q \to \tilde{v} = \tilde{u}$).

As an example, the previously obtained auto-Bäcklund transformation may be used to construct an approximate soliton solution. Starting from $v = 0$, one finds the corresponding function q and spectral parameter λ, which are of the simplest form

$$\lambda = -k^2/4$$

$$q \simeq ((1 - \exp\theta)/(1 + \exp\theta))(24k + \varepsilon k^3(\alpha - 2\beta + \gamma + 30\delta))/48$$

$$\theta = kx + (-k^3 - \varepsilon\delta k^5)t$$

(k is a wavenumber) at

$$96a + k^4(\alpha - 2\beta + \gamma + 30\delta) = 0 \qquad c = 0.$$

The change (2.8) then yields the approximate one-soliton solutions

$$u_{1S} \simeq \frac{k^2}{2}\operatorname{sech}^2\frac{\theta}{2} + \varepsilon\frac{k^4}{48}\left((90\delta - 6\beta - \alpha - 3\gamma)\operatorname{sech}^4\frac{\theta}{2} + 2(\alpha + \gamma + 4\beta - 30\delta)\operatorname{sech}^2\frac{\theta}{2}\right)$$

which were also found by means of the direct perturbation method in [8]. Some of these solutions and their first corrections are depicted in figure 1.

It is interesting to consider this transformation as applied to an N-soliton interaction. By way of illustration, we cite results for a two-soliton solution.

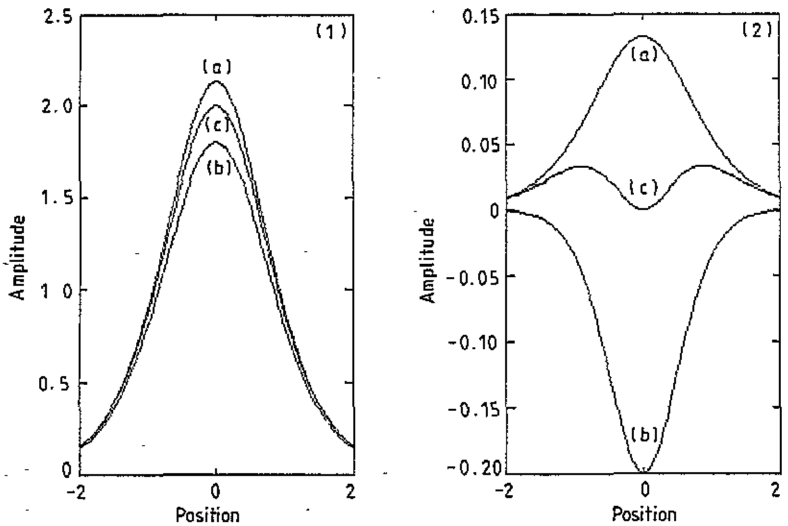

Figure 1. Plot of the PKdV soliton solution with $k = 2$ $\varepsilon = 0.1$ (1), and its first correction (2). (a) $\alpha = 0$; $\gamma, \beta = 1$; $\delta = 0.1$. (b) $\alpha = 0$; $\gamma = 1$; $\delta = -0.1$; $\beta = -1$. (c) $\alpha, \gamma = 1$; $\delta, \beta = 0$.

Starting from the previously derived soliton with the wavenumber k_1 and setting $\lambda=-k_2^2/4$, one arrives at the following expression

$$u_{2S}\simeq 2k_2^2(X-Y^2)+(\varepsilon/6)(2k_2^4(90\delta-6\beta-\alpha-3\gamma)(X^2-2XY^2+Y^4)$$
$$+(\alpha+4\beta+\gamma-30\delta)(2k_1^3k_2(1-Y-Z)$$
$$+k_1^2k_2^2(3X+4Y^2+4YZ-11Y-4Z+4)$$
$$+k_1k_2^3(-8XY+5X+12Y^2+4YZ-11Y-2Z+2)$$
$$+k_2^4(3X^2-8XY+3X+4Y^2-2Y))$$

where

$$Z=(1+\exp\theta_1+\exp\theta_2+A_{12}\exp(\theta_1+\theta_2))^{-1}$$
$$X=Z(k_1^2\exp\theta_1+k_2^2\exp\theta_2+A_{12}(k_1+k_2)^2\exp(\theta_1+\theta_1))/k_2^2$$
$$Y=Z(k_1\exp\theta_1+k_2\exp\theta_2+A_{12}(k_1+k_2)\exp(\theta_1+\theta_1))/k_2$$
$$A_{12}=(k_1-k_2)^2/(k_1+k_2)^2.$$

As expected, at $t\to\infty$ two solitons are completely separated from each other and are described by the expressions obtained above.

It should be stressed that these results cover the cases when the above Lax pair is of the type different from that of the unperturbed one as well as the case ($\beta=10\delta$) when (2.7) possesses the simplest invariance $q\to-q$. Nevertheless, we have the same behaviour of the solution, and the distinctions are only in the numerical coefficients. The results presented in [8, 10] confirm this conclusion as well.

Scalar APs can also be applied to derive recursion formulas for approximate conservation laws. The net result can be presented by

$$q_x=q^2+\lambda q+v+\varepsilon(2(\alpha+2\beta+\gamma-10\delta)(q^4+2\lambda q^3+2q^2v)$$
$$+2\lambda^2(\beta-10\delta)q^2+3\lambda^4\delta+2(-3\beta+\gamma+10\delta)v^2)/12 \tag{2.12}$$

$$q_t=-\frac{\partial}{\partial x}(\lambda^2q+2qv+\lambda v+v_x)+\varepsilon\frac{\partial}{\partial x}(2(-\alpha-2\beta-\gamma+10\delta)q^2v_x+2\lambda^2(-\beta+4\delta)qv$$
$$+2(\alpha+2\beta-8\delta)qv^2-12\delta qv_{xx}-6\delta\lambda^3v-6\delta\lambda^2v_x+\lambda(-\gamma+2\delta)v^2$$
$$-6\delta\lambda v_{xx}+(-\gamma+2\delta)vv_x-6\delta v_{xxx})/6. \tag{2.13}$$

The procedure for finding conservation laws and adiabatic invariants is fully analogous to that described in [13] for the KdV equation and was discussed in [2]. Presenting q as

$$q(x,t)=q_0(x,t)+\varepsilon q_1(x,t)+O(\varepsilon^2) \tag{2.14}$$

and the functions q_0, q_1 as

$$q_0(x,t)=\sum_{i=1}^{\infty}\lambda^{-i}W_{0i}(x,t)$$

$$q_1(x,t)=\sum_{j=-3}^{\infty}\lambda^{-j}W_{1j}(x,t)$$

($W_{0i}(x, t)$ and $W_{1j}(x, t)$ are new functions) then substituting (2.14) into (2.12) and (2.13), omitting the terms of $O(\varepsilon^2)$ order, and making the coefficients at the powers of λ equal to zero, from (2.12) one obtains the recurrence formulas to determine W_{0i} and W_{1i} with respect to W_{0j} and W_{1j} $(j<i)$. Accordingly equation (2.13) determines approximate conservation laws. The corresponding adiabatic invariants are of the form

$$\rho_j=\int_{-\infty}^{+\infty} W_{0j}\,\mathrm{d}x+\varepsilon\int_{-\infty}^{+\infty} W_{1j}\,\mathrm{d}x \qquad j=\overline{1,\infty} \tag{2.15}$$

and some of them are presented in appendix 2.

In conclusion, it should be stressed that the scattering problem, auto-Bäcklund transformation, and recursion formulas involved cannot be derived from those for the higher-order KdV equation by means of the approximate canonical transformation [10].

3. A two-dimensional spectral problem for an extended version of the PKdV equation

Additional results may be given for an equation of the type [16]

$$u_t+u_{xxx}+6uu_x+\varepsilon(\xi u_{xxx}u_{xx}+\delta u_{xxx}u^2+\varkappa u_{xxx}u+\gamma u_{xx}u_xu+\rho u_{xx}u_x+\zeta u_{xxxx}u_x$$
$$+\sigma u_{xxxxx}u+\eta u_{xxxxx}+\mu u_{xxxxxxx}+\beta u_x^3+\alpha u_xu^3+\nu u_xu^2)=0. \tag{3.1}$$

Here we confine ourselves to APs consistent with $sl(2)$ Lie-algebra [9], because the two-dimensional scattering problem can be described in terms of the basis $\{E, F, H\}$ with the commutation relations [9]

$$[E,E]=0 \qquad [F,F]=0 \qquad [H,H]=0$$
$$[H,E]=2E \qquad [H,F]=-2F \qquad [E,F]=H.$$

The elements have the following two-dimensional linear representation

$$E=\begin{pmatrix}0&1\\0&0\end{pmatrix}q \qquad F=\begin{pmatrix}0&0\\1&0\end{pmatrix}q \qquad H=\begin{pmatrix}1&0\\0&-1\end{pmatrix}q \tag{3.2}$$

and the nonlinear one-dimensional representation (2.2) [17]. (Matrix unit is not essential for scattering problems, because it commutes with any matrices and so adds just terms corresponding to conservation laws.)

The resulting expressions for the equation determining the evolution of the function q with respect to x is written as

$$q_x=(v+a)E+bH-F+\varepsilon(C_eE+C_fF+C_hH)/36 \tag{3.3}$$

$$C_e=(-392\mu+3\alpha+12\beta-3\gamma-20\delta+98\sigma)v^3$$
$$+3a(-392\mu+3\alpha+12\beta-5\gamma-8\delta+78\sigma)v^2$$
$$+2b^2(560\mu-3\alpha-24\beta+9\gamma+8\delta-110\sigma)v^2+6(\rho-3\varkappa+10\eta)v^2$$
$$4a^2(-308\mu+3\alpha+6\beta-3\gamma-2\delta+32\sigma)v$$
$$+4ab^2(644\mu-3\alpha-6\beta+3\gamma+2\delta-56\sigma)v+6a(-\nu+\rho-2\varkappa+30\eta)v$$
$$+96b^4(-14\mu+\sigma)v+24b^2(\varkappa-10\eta)v+72bcv+36hv+36d$$

$$C_f=(56\mu-3\alpha+12\beta-3\gamma+8\delta-14\sigma)v^2+4a(28\mu+3\alpha+6\beta-3\gamma-2\delta+8\sigma)v$$
$$+4b^2(-28\mu-3\alpha-6\beta+3\gamma+2\delta-8\sigma)v+6(-\nu+\rho+2\varkappa-10\eta)v+36h$$
$$C_h=b(-56\mu+3\alpha-12\beta+3\gamma-8\delta+14\sigma)v^2+36cv+36g$$
$$a, b, c, d, h, g=\text{constant.}$$

In addition, for the AP to be consistent with $sl(2)$, it is necessary that the following relations hold for the parameters of (3.1)

$$\zeta=(224\mu-3\alpha-24\beta+9\gamma+8\delta-50\sigma)/12$$
$$\xi=(280\mu-3\alpha-24\beta+9\gamma+8\delta-70\sigma)/4.$$

An auxiliary time evolution problem is trivially restored from (3.3) [13] and not presented here because of complexity.

In the case of a linear AP (see (3.2)) (3.3) corresponds to the AKNS representation of a scattering problem if $b=\lambda$ (λ is the spectral parameter) and $a=0$. c, d, h, g may be chosen in such a way that (3.3) is reduced to some investigated spectral problem (for example, a quadratic [18] or an arbitrary polynomial bundle [19]). Here we do not consider this question.

On the other hand, using the nonlinear representation (2.2), auto-Bäcklund transformations and a Lax pair can also be derived in the same way as in section 2. For instance, letting $b=0$ and $a=\lambda$, one arrives at the AP that is of the form of a Riccati equation and corresponds to the Lax pair. And the simplest auto-Bäcklund transformation corresponding to the trivial invariance $q\to -q$ takes place at

$$\varkappa=10\eta \qquad \alpha=-8\beta+3\gamma \qquad \sigma, \delta, \mu, \xi, \zeta=0.$$

In so doing we have the relation

$$q_x=v+q^2+\lambda+\varepsilon\big((-2\beta+\gamma)v^3+2v^2(q^2+\lambda)(-3\beta+\gamma)+(\rho-20\eta)v^2$$
$$+4\lambda(3\beta-\gamma)vq^2+(\nu-\rho-10\eta)vq^2+4\lambda^2(-3\beta+\gamma)v$$
$$+\lambda(-\nu+\rho+10\eta)v+6hv-6hq^2+6d\big)/6$$

and the following equation for q

$$q_t=6q^2q_x+6\lambda q_x-q_{xxx}+\varepsilon\big(4(-3\beta+\gamma)q^6q_x+2(-24\lambda\beta+8\lambda\gamma-\nu+\rho-5\eta)q^4q_x$$
$$4(3\beta-\gamma)q^3q_xq_{xx}+2(-6h-18\lambda^2\beta+6\lambda^2\gamma-\lambda\nu+\lambda\rho-20\lambda\eta)q^2q_x$$
$$+10\eta q^2q_{xxx}+(24\lambda\beta-8\lambda\gamma+\nu-\rho+30\eta)qq_xq_{xx}+10\eta q_x^3$$
$$+(-3\beta+\gamma)q_xq_{xx}^2+6(d-h\lambda-5\lambda^2\eta)q_x+10\lambda\eta q_{xxx}-\eta q_{xxxxx}\big)+O(\varepsilon^2).$$

The transformations involved can be used to construct first-order corrections to the N-soliton solutions of the KdV equation.

4. An approximate scattering problem and auto-Bäcklund transformation of the perturbed MKdV equation

In the previous sections the APs of the first order for the perturbed KdV equation have been considered in detail. This perturbed nonlinear equation contains the terms with the small parameter that are presented in the higher-order KdV equation [9]. From the viewpoint of theory and practice [20, 21] it is interesting to consider APs for the MKdV equation with the terms that are presented in a higher-order MKdV equation [9]

$$u_t+6u^2u_x+u_{xxx}+\varepsilon(\beta u^2u_{xxx}+\gamma uu_xu_{xx}+\delta u_x^3+\alpha u_{xxxxx}+\zeta u^4u_x)=0. \tag{4.1}$$

In order to construct an AP associating with the inverse scattering problem and auto-Bäcklund transformation, let us choose P_0 and Q_0 corresponding to the ones of the MKdV equation [9]

$$\begin{aligned}&P_0=vE-vF+\lambda H\\&Q_0=(-2v^3-4\lambda^2v-2\lambda v_x-v_{xx})E+(2v^3+4\lambda^2v-2\lambda v_x+v_{xx})F-2\lambda(v^2+2\lambda^2)H.\end{aligned} \tag{4.2}$$

Proceeding in the same manner as shown above, we can find the form of P_1, Q_1 and corresponding commutation relations. Since their detailed analysis involves difficulties, we mention only that P_1 and Q_1 can be written as follows

$$\begin{aligned}&P_1=Av^2+Bv+J\\&Q_1=G(v,\ldots,v_{xxxx};A,B,J,E,F,H)+D\end{aligned} \tag{4.3}$$

where A, B, J, D together with E, F, H belong to some Lie-algebra. As in section 3, they can be expressed in terms of E, F, H for our purposes, and for P_1 we finally obtain

$$\begin{aligned}P_1=((6av+6b+(-2\beta+\gamma-2\delta)v^3-80\lambda^2\alpha v+4\lambda^2(\beta+\gamma-3\delta)v)E\\+(6av-12d\lambda+6b+(2\beta-\gamma+2\delta)v^3)F\\+2(3dv+3c+(-10\lambda\alpha+\lambda\gamma-3\lambda\delta)v^2)H)/6.\end{aligned}$$

(Q_1 is presented in appendix 3). In addition, the commutation relations impose a restriction on the parameters of (4.1):

$$\zeta=-20\alpha+\beta+2\gamma-4\delta.$$

The two-dimensional eigenvalue problem is directly determined by the formula

$$q_x=P_0+\varepsilon P_1$$

if we use the representation (3.2). On the other hand, the nonlinear representation (2.2) for the basis of $sl(2)$ results in the Bäcklund transformation mapping equation (4.1) into an equation that is invariant with respect to $\lambda\to-\lambda$ when $b, c=0$; $\beta=10\alpha$, and the AP can be employed as an auto-Bäcklund transformation by analogy with the corresponding pseudopotential of the MKdV equation [22].

First, let us derive exact pseudopotentials corresponding to the equations for an MKdV solution and next correction (see section 1). Replacing q and v in (4.2) and (4.3) by the formal expansions

$$q=q_0(x,t_0,t_1,\ldots)+\varepsilon q_1(x,t_0,t_1\ldots)+\ldots$$

$$v=v_0(x,t_0,t_1\ldots)+\varepsilon v_1(x,t_0,t_1\ldots)+\ldots$$

using the method of multiple scales

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial t_0}+\varepsilon\frac{\partial}{\partial t_1}+\ldots\qquad t_i=\varepsilon^i t$$

and equating each coefficients in ε to zero, one obtains the pseudopotentials for v_0 and v_1. It should be stressed that at this stage in the expansion for q ε plays the role of a graduating parameter (as for instance in Hirota's method [9, 13]).

Then starting from $v=0$ and letting $a=\lambda^2(10\alpha-\gamma+3\delta)/3$, one has

$$q_0=\exp\theta \qquad \theta=2\lambda x+8\lambda^3 t_0+32\alpha\lambda^5 t_1$$

$$q_1=C\exp(2\theta) \qquad C=\text{constant}$$

and although these functions are unbounded, the exact pseudopotentials involved result in correct expressions for v_0 and v_1. Replacing $\lambda\to-\lambda$, one finally obtains new v_0, v_1 and the corresponding approximate one-soliton solution of (4.1)

$$u\simeq-2\lambda\operatorname{sech}\theta+2/3\varepsilon\lambda^3((-10\alpha+\gamma-3\delta)\operatorname{sech}\theta-2(10\alpha-\delta)\operatorname{sech}^3\theta).$$

Now let us dwell on a particular case of equation (4.1). At $\beta, \gamma, \delta=0$; $\alpha=1$ one has the equation

$$u_t+6u^2u_x+u_{xxx}+\varepsilon(-20u^4u_x+u_{xxxxx})=0 \tag{4.4}$$

which is a special case of

$$u_t+f(u)u_x+u_{xxx}+\varepsilon u_{xxxxx}=0. \tag{4.5}$$

The latter may be considered as a generalized Kawahara equation [23]. Equation (4.4) is derived from (4.5) as the leading order approximation in an appropriate asymptotic sense under the assumption of weak nonlinearity.

The addend $-20\varepsilon u^4u_x$ in (4.4) can be shown to be necessary for the existence of the AP leading to the scattering problem. For this purpose it is sufficient to use just two commutation relations

$$[E,A]-[F,A]-(20/3)\lambda(E+F)=0 \tag{4.6}$$

$$[E,[E,[F-E,A]]]+[E,[F,[E-F,A]]]+4[F-E,A]+[F,[E,[E-F,A]]]$$
$$+[F,[F,[F-E,A]]]-32\lambda(E+F)=0 \tag{4.7}$$

which are obtained by substituting P_1, Q_1 into (1.6) and equating the coefficients at v_{xx} and v^5 to zero.

From (4.6) one obtains

$$[E,A]=[F,A]+20\lambda(E+F)/3.$$

Inserting this expression into (4.7), one has

$$\lambda(E+F)=0 \qquad \lambda\neq0.$$

That is (4.6) and (4.7) are not consistent. Thus, for the equation

$$u_t+6u^2u_x+u_{xxx}+\varepsilon u_{xxxxx}=0$$

there exists no AP of the first kind [3] in any dimension that prolongs the MKdV scattering problem.

5. Conclusions

It is necessary to stress that, although alternative techniques can be used to obtain approximate auto-Bäcklund transformations, conservation laws, etc. (e.g. the Lax pair approach instead of pseudopotentials or direct methods to derive one-soliton solutions), they are not applicable in many cases.

In the case of the PKdV equation, for example, the spectral problem of the Lax pair depends on v and v_x, while the related AP depends only on v. As a result, we have solved the ordinary differential equations for finding its form, while the presence of v_x or any higher derivatives would lead to a set of PDEs. The latter could not be solved in a general case. Hence the result has been obtained in the simpler manner, although the final expressions are, of course, the same.

In the case of the perturbed MKdV, the use of the Lax representation would result in highly cumbersome computations. In addition, this representation is not typical for such equations.

One more limitation of the Lax approach is associated with initial assumptions about the order of a spectral operator. On the other hand, in the framework of the AP approach this problem is solved by closing a Lie algebra and finding its representation. For instance, in section 4 it has been shown that there is no scattering problem (related to the simplest AP of the first kind) for some parameters. In other words, there is no related Lax pair with an operator of any order as well.

Moreover, frequently it is necessary to investigate special type equations and to prolong, for example, linearizing transformations or a finite number of conservation laws. In such cases APs are the only applicable technique. Finally, APs allow one to derive spectral problems, auto-Bäcklund transformations, and conservation laws within the framework of a unified approach. All the results mentioned above are related to each other, because they are associated with the same APs or their special cases.

Next it is important to point out two problems, which are still to be solved.

First, in order to make the method fully rigorous, it is necessary to show that

$$|u(x, t; \varepsilon) - v(x, t; \varepsilon)| = o(\varepsilon^m) \text{ for } \varepsilon \to 0$$

as well as to know the time domain in which this is indeed valid. This is actual both for approximate Bäcklund transformations and for scattering problems [24–26]. For example, although the scattering data appear to be only slightly perturbed on the time-scale under consideration, one cannot yet conclude that the same holds for $u(x, t)$.

Second, as in the case of exact pseudopotentials, the existence of APs is a consequence of very special balance between the various terms in an equation and not always takes place as shown in section 4. At the same time, as shown above for the perturbed KdV, MKdV and perturbed NLS [27] equations, there exist approximately 'integrable' perturbations associated to Lax pairs or related APs (S-integrable according to F Calogero [35]). On the one hand, perturbed equations that are of great interest in physics and at the same time admit approximate scattering problems can be sequentially considered and classified. On the other hand, it is important to propose criteria or signs which could characterize them. Investigations in this field have been carried out in the last few years [14, 15, 27, 28]. In [27] this attempt was undertaken on the basis of the hypotheses on the Painleve property [29, 30] and the existence of Lie–Bäcklund (non-Lie-point) symmetries [31]. It collapsed, because the original definitions were used. In [14, 28] the theory of approximate symmetries was established. The obtained approximate Lie–Bäcklund ones of the PKdV indicate the existence of a Lax pair. Recently in [15], a generalization of the Painleve approach was proposed and modified for perturbed PDEs. These late results also indicate such integrability. In particular, the extra constraint $\zeta = -20\alpha + \beta + 2\gamma - 4\delta$ for (4.1) was obtained.

In the future it would be of interest to consider their use for higher order perturbations and derived so-called resonance conditions [10, 32] on the parameters of perturbations.

All these problems are, of course, beyond the scope of the present work and require special study and review (see also [33, 34]).

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Appendix 1

In the case of equation (2.1) for P_1 and Q_1 associated with

$$P_0=(\lambda+v)E-F$$

$$Q_0=(4\lambda^2+2\lambda v-2v^2-v_{xx})E+2(v-2\lambda)F-v_xH$$

one has, using (1.6), the equation

$$-(-\alpha v^2v_x+\beta vv_{xxx}+\gamma v_xv_{xx}+\delta v_{xxxxx})\frac{\partial}{\partial v}P_0-(6vv_x+v_{xxx})\frac{\partial}{\partial v}P_1+[P_0, Q_1]+[P_1, Q_0]$$

$$-\left(v_{xxxxx}\frac{\partial}{\partial v_{xxxx}}+v_{xxxx}\frac{\partial}{\partial v_{xxx}}+v_{xxx}\frac{\partial}{\partial v_{xx}}+v_{xx}\frac{\partial}{\partial v_x}+v_x\frac{\partial}{\partial v}\right)Q_1=0.$$

Solving it as shown in [4], one obtains

$$P_1=v^2A+vB+J \tag{A1.1}$$

$$\begin{aligned}Q_1=\lambda^2v(-[E,[E,B]]-8\delta E)+\lambda v^2(-\tfrac{1}{2}[E,[E,B]]+(\gamma-14\delta)E/3\\
+\lambda v([E,[F,B]]-[E,[J,E]]+[F,[E,B]]-8\delta F)+\lambda v_x(-[E,B]+4\delta H)\\
+2\lambda\delta v_{xx}E+v^3(\tfrac{1}{3}[H,A]-4A+(3\alpha+2\gamma-4\delta)E/9)\\
+v^2(\tfrac{1}{2}[E,[F,B]]-\tfrac{1}{2}[E,[J,E]]+\tfrac{1}{2}[H,B]-3B+(\gamma-2\delta)F/3)\\
+vv_x(-\gamma+2\delta)H/3+vv_{xx}(-2A+(2\delta-\beta)E)\\
+v(-[F,[F,B]]+[F,[J,E]]-[J,H])+v_x^2(A+(\beta-\gamma-2\delta)E/2)\\
+v_x([F,B]-[J,E])+v_{xx}(2\delta F-B)-\delta v_{xxx}H-\delta v_{xxxx}E+D\end{aligned}$$

(A, B, J, D are new functions of q) and additional conditions that are of the form

$$6[F,A]+(-3\beta+\gamma+10\delta)H=0 \qquad [E,A]=0 \qquad [E,[H,A]]=0$$

$$\begin{aligned}4/6(3\alpha+9\beta+2\gamma-40\delta)H-3\lambda[E,[E,[E,B]]]+3[E,[E,[F,B]]]\\
-3[E,[E,[J,E]]]+3[E,[H,B]]-6[E,B]-2[F,[H,A]]=0\end{aligned}$$

$$\begin{aligned}4\lambda(\beta-10\delta)H-3\lambda^2[E,[E,[E,B]]]+3\lambda[E,[E,[F,B]]]-3\lambda[E,[E[J,E]]]\\
+2\lambda[E,[F,[E,B]]]-2[E,[F,[F,B]]]+2[E,[F,[J,E]]]\\
+\lambda[E,[H,B]]-2[E,[J,H]]-10\lambda[E,B]+\lambda[F,[E,[E,B]]]\\
-[F,[E,[F,B]]]+[F,[E,[J,E]]]-[F,[H,B]]+2[F,B]-4[J,E]=0\end{aligned}$$

$$-16\lambda^2\delta H-\lambda^3[E,[E,[E,B]]]+\lambda^2[E,[E,[F,B]]]-\lambda^2[E,[E,[J,E]]]$$
$$+\lambda^2[E,[F,[E,B]]]-\lambda[E,[F,[F,B]]]+\lambda[E,[F,[J,E]]]-\lambda[E,[J,H]]$$
$$-4\lambda^2[E,B]+[E,D]+\lambda^2[F,[E,[E,B]]]-\lambda[F,[E,[F,B]]]$$
$$+\lambda[F,[E,[J,E]]]-\lambda[F,[F,[E,B]]]+[F,[F,[F,B]]]-[F,[F,[J,E]]]$$
$$+[F,[J,H]]+4\lambda[F,B]+2\lambda[J,E]+2[J,F]=0$$
$$\lambda([E,D]+4[J,E]-4[J,F])-[F,D]=0.$$

These relations have a particular solution in the form of polynomials of the second degree

$$A=(-3\beta+\gamma+10\delta)/6 \qquad B=q^2(-\alpha-2\beta-\gamma+10\delta)/6+hq-c$$
$$J=\lambda q^2(\alpha-2\beta+\gamma+30\delta)/6+q^2c+\lambda hq+bq-\lambda c+a$$
$$D=(4\lambda^2q^2(\alpha-2\beta+\gamma+18\delta)+12\lambda cq^2+12aq^2+12\lambda^2hq+12\lambda bq+2\lambda^3(\alpha-2\beta+\gamma+6\delta)$$
$$-12\lambda^2c+24\lambda a)/3 \qquad a,b,c,h=\text{constant.} \tag{A1.2}$$

Appendix 2

Here we present the first four non-trivial adiabatic invariants (2.15) obtained via the recursion relations (2.12) and (2.13). The corresponding expressions are as follows

$$\rho_1=\int_{-\infty}^{+\infty}u\,dx \qquad \rho_3=\int_{-\infty}^{+\infty}-6u^2\,dx+\varepsilon(2\beta-\gamma)\int_{-\infty}^{+\infty}u_x^2\,dx$$
$$\rho_5=\int_{-\infty}^{+\infty}(60u^3+180uu_{xx}+150u_x^2)\,dx-\varepsilon\int_{-\infty}^{+\infty}(10u^2u_{xx}(-\alpha-14\beta+8\gamma-50\delta)$$
$$+10uu_x^2(-\alpha-19\beta+13\gamma-100\delta)+u_{xx}^2(-\alpha-10\beta+5\gamma-30\delta))\,dx$$
$$\rho_7=\int_{-\infty}^{+\infty}(210u^4+1260u^2u_{xx}+2100uu_x^2+756u_xu_{xxx}+798u_{xx}^2)\,dx$$
$$-\varepsilon\int_{-\infty}^{+\infty}(28u^3u_{xx}(-4\alpha-33\beta+16\gamma-110\delta)$$
$$+14u^2u_x^2(-14\alpha-138\beta+81\gamma-660\delta)$$
$$+84uu_xu_{xxx}(-6\alpha-27\beta+13\gamma-170\delta)$$
$$+14uu_{xx}^2(-39\alpha-178\beta+83\gamma-1050\delta)$$
$$+2u_x^2u_{xx}(-425\alpha-1824\beta+931\gamma-13130\delta)$$
$$+u_{xxx}^2(3\alpha+15\beta-7\gamma+80\delta))\,dx$$

(the function v has been replaced by u for clearness).

Appendix 3

Here we present the expression for Q_1 (4.3) associated with P_0 and Q_0 (4.2)

$$Q_1=(C_eE+C_fF+C_hH)/6$$

and

$$\begin{aligned}C_e=&16\lambda^4(-\beta-\gamma+3\delta+14\alpha)v+8\lambda^3(-\beta-\gamma+3\delta+14\alpha)v_x\\&+4\lambda^2((-2\beta-3\gamma+8\delta+48\alpha)v^3+(-\beta-\gamma+3\delta+4\alpha)v_{xx}-6av-6b)\\&+2\lambda((4\alpha-\gamma)v^2v_x-6\alpha v_{xxx}-6av_x-24cv)+3(2(\beta-\gamma+2\delta+4\alpha)v^5\\&+(-\gamma+2\delta)v^2v_{xx}-2\alpha v_{xxxx}-4av^3-2av_{xx}-4bv^2-4cv_x)\\C_f=&96\lambda^4\alpha v+48\lambda^3(d-\alpha v_x)+4\lambda^2((\gamma-2\delta-8\alpha)v^3+6\alpha v_{xx}-6av-6b)\\&+2\lambda((-\gamma+4\alpha)v^2v_x-6\alpha v_{xxx}+6av_x+12dv^2+24cv))\\&+3(2(\gamma-\beta-2\delta-4\alpha)v^5+(\gamma-2\delta)v^2v_{xx}\\&+2\delta vv_x^2+2\alpha v_{xxxx}-4av^3-2av_{xx}-4bv^2-4cv_x)\\C_h=&-96\lambda^5\alpha+8\lambda^3(-\beta-\gamma+3\delta+14\alpha)v^2-24\lambda^2(dv+3c)\\&+2\lambda((\beta-4\gamma+10\delta+32\alpha)v^4+(-2\gamma+8\alpha+6\delta)vv_{xx}\\&+(\gamma-3\delta-4\alpha)v_x^2-6dv_x)+6(-2dv^3-dv_{xx}+2bv_x-2cv^2).\end{aligned}$$

P_0 is presented in section 4.

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